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LETTER TO THE EDITOR

The interaction of the ω-soliton and ω-cuspon of the Camassa–Holm equation

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Abstract
In this letter, we use the newly available explicit multi-soliton and multi-cuspon solutions of the Camassa–Holm equation to study the interactions of a soliton and a cuspon, two cuspons and two solitons. Some interesting phenomena are found, e.g., a larger soliton can 'eat up' a smaller cuspon during the collison and it is the other way round if the amplitude of the cuspon is larger. It is also found that a soliton and a cuspon can emerge from an almost zero mass. The interaction of two cuspons is also investigated in detail for the first time.

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1. Introduction

The Camassa–Holm equation

$$\partial_t u + 2\omega\partial_x u - \partial_{x,x,t} u + 3u\partial_x u - 2\partial_x u\partial_{x,x} u - u\partial_{x,x,x} u = 0 \tag{1.1}$$

was proposed in Camassa and Holm (1993) and Camassa *et al* (1994) as a model equation for unidirectional nonlinear dispersive waves in a shallow water. Johnson (2002) questioned the validity of the first derivation of (1.1) given in the paper by Camassa and Holm (1993) and provided a consistent derivation for (1.1) as a model equation in a shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties, e.g., it is an integrable equation and admits the peakon solution. In the context of another valid physical model, Dai (1998a) (see also Dai and Huo 2000) derived the following model equation for nonlinear dispersive waves in cylindrical hyperelastic rods:

$$\partial_t v + 3v\partial_x v - \partial_{x,x,t} v - \gamma(2\partial_x v\partial_{x,x} v + v\partial_{x,x,x} v) = 0, \tag{1.2}$$

where γ is a material parameter. In Dai and Huo (2000) the phase plane technique was used to analyse the travelling wave solutions. A variety of analytical solutions were obtained for γ

within five different intervals. Constantin and Strauss (2000a) proved that the solitary waves of this model are orbitally stable for the case of $\gamma \leqslant 1$; and for the case of $\gamma = 1$ they (Constantin and Strauss 2000b) also gave the proof of the orbital stability of the peakons in the H^1 norm. These important stability results imply that when two solitons are well apart (e.g, the case of two-soliton solution for large t), the influence of one soliton on another is negligible. Recently, Ivanov (2005) has shown that (1.2) is integrable if and only if $\gamma = 1$. Obviously, equation (1.2) is reduced to equation (1.1) for the case of $\omega = 0$ when $\gamma = 1$, i.e.,

$$\partial_t U + 3U\partial_x U - \partial_{x,x,t} U - 2\partial_x U \partial_{x,x} U - U\partial_{x,x,x} U = 0. \tag{1.3}$$

We note that one can get the solution of equation (1.3) from the solution of equation (1.1) as below

$$U(x,t) = u(x - \omega t, t) + \omega. \tag{1.4}$$

Besides admitting the peakon solution, equation (1.1) also has the so-called cuspon solution.

In Ferreira *et al* (1999), they investigated the interaction of a soliton and a cuspon with the help of numerical methods for the case of $\omega = 2$, as no explicit analytical expressions were avaliable at the time. Nevertheless, these authors managed to obtain some analytical results for the phase shifts after the interaction. Recently, important progress has been made to obtain the explicit multiple-soliton solutions of the Camassa–Holm equation. Constantin (2001) made a connection between the spectral problem of the Camassa–Holm equation and that of the KdV equation. Johnson (2003) implemented Constantin's scattering approach and obtained the two-soliton solution. However, Johnson's approach was difficult to yield multiple-soliton (say,three-soliton) solution. Parker (2004) managed to obtain the associated bilinear form of the Camassa–Holm equation and analysed the one soliton (the one-soliton solution was already previously obtained by Dai 1998b by the phase-plane technique). Although Parker pointed out that his bilinear form approach could yield the multiple-soliton solution, the results are still not published. In Li and Zhang (2004) and Li (2005), a different approach associated with the Darboux transformation was introduced to construct the explicit expressions for the multiple-soliton solutions. The results are summarized below.

Denote the two fundamental solutions of the KdV spectral problem with a zero potential as

$$\Phi_i = \begin{cases} \cosh \xi_i & i \text{ is odd,} \\ \sinh \xi_i & i \text{ is even,} \end{cases}$$

$$\xi_i = k_i \left(y + \frac{\sqrt{\omega} t}{2\left(k_i{}^2 - \frac{1}{4\omega}\right)} \right), \qquad i = 1, 2, \dots n. \tag{1.5}$$

Then, the n-soliton solution of the Camassa–Holm equation (1.1) is given by

$$u(y,t) = \partial_t \ln \left(\frac{f_1}{f_2} \right), \tag{1.6}$$

$$f_1 = \frac{Wx\left(\Phi_1, \Phi_2, \dots \Phi_n, \mathrm{e}^{\frac{y}{2\sqrt{\omega}}}\right)}{W(\Phi_1, \Phi_2, \dots \Phi_n)}, \qquad f_2 = \frac{W\left(\Phi_1, \Phi_2, \dots \Phi_n, \mathrm{e}^{\frac{-y}{2\sqrt{\omega}}}\right)}{W(\Phi_1, \Phi_2, \dots \Phi_n)}, \tag{1.7}$$

where $W(\Phi_1, \Phi_2, \dots \Phi_n)$ is the Wronskian and the parameter y is realated to x through

$$x = \ln \left(\sqrt{\frac{f_1^2}{f_2^2}} \right). \tag{1.8}$$

This approach can be extended to $\Phi_1 = \sinh \xi_1$, or $\Phi_1 = \cosh \xi_1$, $\Phi_2 = \cosh \xi_2$, etc.

For the above solutions, even when $n = 2$, a few cases can arise. In this letter we shall examine in detail the interactions and an ω-soliton and an ω-cuspon, two ω-solitons and two ω-cuspons. Our investigation based on the explicit solutions reveals some interesting phenomena, e.g., when the amplitude of the ω-soliton is larger than that of the ω-cuspon, the soliton actually 'eats up' the cuspon during the interaction process and it is the other way round if the amplitude of the ω-cuspon is larger.

The main purpose of this letter is to use explicit solution expressions to describe the interaction processess and to report some phenomena which have not been found by numerical studies (Camassa and Holm 1994, Ferreira *et al* 1999).

2. One ω-soliton solution and one ω-cuspon solution

When two solitons or two cuspons are well separated, each of them can be regarded as a single soliton or a single cuspon. So, in this section we first represent the solutions of an ω-soliton and an ω-cuspon.

2.1. ω-soliton

In equation (1.5), we take $n = 1$ and $\Phi_1 = \cosh \xi_1$. Then, we have

$$f_1 = \frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{w}\tanh \xi_1)}{2\sqrt{\omega}},$$

$$f_2 = \frac{\exp\left(\frac{-y}{2\sqrt{\omega}}\right)(1 + 2k_1\sqrt{\omega}\tanh \xi_1)}{2\sqrt{\omega}},$$

$$u(y, t) = \partial_t \ln\left(\frac{f_1}{f_2}\right) = \frac{8k_1\omega^2}{\left(4k_1{}^2\omega - 1\right)\left(4k_1{}^2\omega \sinh \xi_1{}^2 - \cosh \xi_1{}^2\right)},$$

$$x = \ln\left(\frac{-f_1}{f_2}\right) = \ln\left(\frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{\omega}\tanh \xi_1)}{1 + 2k_1\sqrt{\omega}\tanh \xi_1}\right).$$

When $2k_1\sqrt{\omega} < 1$, y and x are in one-to-one correspondence. We set

$$k_1 = \frac{\sqrt{1 - \frac{2\omega}{c_1}}}{2\sqrt{\omega}}.$$

Functions $u(x, t)$ and $\partial_x u(x, t)$ are continuous functions of x and t and further $u(x, t) > 0$. Also, as $|x| \to \infty, u \to 0$. The relation between the amplitude h and velocity c_1 reads as $h = c_1 - 2\omega$. For this solution $u(x, t)$ contains two parameters: ω and the velocity c_1; and we call this solution $u(x, t)$ as an ω-soliton. In Parker (2004), the author pointed out that when both $\omega \to 0$ and $2k_1\sqrt{\omega} \to 1$, the solution $u(x, t)$ tends to the peakon solution: $c_1 \exp(-|x - c_1 t|)$. Correspondingly for equation (1.3), $U \to \omega$ as $|x| \to \infty$. The relation between the amplitude H and c_1 for U reads as $H = c_1 - \omega$.

2.2. ω-cuspon

We take $n = 1$ and $\Phi_1 = \sinh \xi_1$. Then, we have

$$f_1 = \frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{\omega}\coth \xi_1)}{2\sqrt{\omega}},$$

$$f_2 = \frac{\exp\left(\frac{-y}{2\sqrt{\omega}}\right)(1 + 2k_1\sqrt{\omega}\coth\xi_1)}{2\sqrt{\omega}},$$
$$u(y,t) = \partial_t \ln\left(\frac{f_1}{f_2}\right) = \frac{8k_1\omega^2}{\left(4{k_1}^2\omega - 1\right)\left(4{k_1}^2\omega\cosh{\xi_1}^2 - \sinh{\xi_1}^2\right)},$$
$$x = \ln\left(\frac{f_1}{f_2}\right) = \ln\left(\frac{\exp\left(\frac{y}{2\sqrt{\omega}}\right)(1 - 2k_1\sqrt{\omega}\coth\xi_1)}{(1 + 2k_1\sqrt{\omega}\coth\xi_1)}\right).$$

When $2k_1\sqrt{\omega} > 1$, y and x are in one-to-one correspondence. We set

$$k_1 = \frac{\sqrt{1 - \frac{2w}{c_1}}}{2\sqrt{w}}.$$

Function $u(x,t)$ is a continuous function of x and t and further $u(x,t) < 0$. However, there is one point at which $\partial_x u(x,t)$ tends to ∞. Further, as $|x| \to \infty$, $u \to 0$. The relation between the amplitude h and velocity c_1 reads as $h = c_1$. Also, the solution $u(x,t)$ contains two parameters: ω and the velocity c_1, and we call this solution as an ω-cuspon. Correspondingly for equation (1.3), when $|x| \to \infty$, $U \to \omega$, and the relation between the amplitute H and c_1 for U reads as $H = c_1 + \omega$.

In the next section, we describe the interactions between solitons and cuspons. We shall take the value of ω to be very small, so that approximately the results also apply to equation (1.3).

3. The interaction processes

3.1. The interaction of two ω-solitons

We take $n = 2$ in equations (1.5)–(1.8) and set $k_1 = \sqrt{1 - \frac{2\omega}{c_1}}\big/(2\sqrt{\omega})$, $k_2 = \sqrt{1 - \frac{2\omega}{c_2}}\big/(2\sqrt{\omega})$. When $c_2 > c_1 > 0$, the corresponding solution is the two-soliton solution.

In figure 1, we describe the interaction process of two solitons for seven different times. Such an interaction is very similar to that of two solitons of the KdV equation, except that the two solitons of the Camassa–Holm equation are never merged into a single hump when the collision happens.

3.2. The interaction of two ω-cuspons

We take $n = 2$ in equations (1.5)–(1.8) and set $k_1 = \sqrt{1 - \frac{2\omega}{c_1}}\big/2\sqrt{\omega}$ and $k_2 = \sqrt{1 - \frac{2\omega}{c_2}}\big/2\sqrt{\omega}$. When $c_2 < c_1 < 0$, the corresponding solution is the two-cuspon solution.

During to the singularities of the two cuspons, their interaction process may be difficult to be investigated by numerical means. Actually, no work has been done to study in detail the interaction of two cuspons of the Camassa–Holm equation. Here, with the explicit solutions, we have no difficulty describing the interaction of two cuspons in detail. Figure 2 shows the interaction process for seven different times. We can see that the interaction has the character similar to that of two solitons. However, we note that the two cuspons (i.e., two points where $\partial_x u \to \infty$) are always present, before, during and after the collision.

3.3. The interaction of one ω-soliton and one ω-cuspon

We take

$$W(\Phi_1, \Phi_2) = W(\cosh\xi_1, \cosh\xi_2),$$

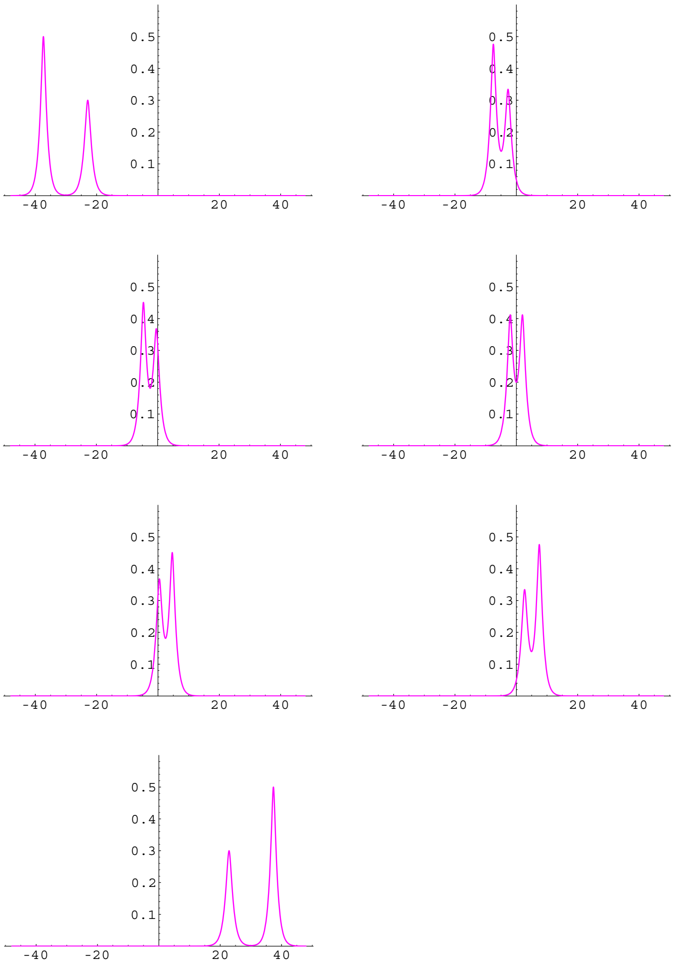

Figure 1. The interaction of two ω-solitons. The velocities of two solitons are respectively $c_1 = 0.4$ and $c_2 = 0.6$. From top to bottom, the time $t = (-60, -10), (-5, 0), (5, 10), 60$.

$$f_1 = \frac{W\left(\cosh\xi_1, \cosh\xi_2, \exp\left(\frac{y}{2\sqrt{\omega}}\right)\right)}{W(\cosh\xi_1, \cosh\xi_2)},$$

$$f_2 = \frac{W\left(\cosh\xi_1, \cosh\xi_2, \exp\left(\frac{-y}{2\sqrt{\omega}}\right)\right)}{W(\cosh\xi_1, \cosh\xi_2)},$$

$$u(y,t) = \partial_t \ln\left(\frac{f_1}{f_2}\right), \qquad x = \ln\left(\sqrt{\frac{f_1^2}{f_2^2}}\right),$$

$$k_1 = \frac{\sqrt{1 - \frac{2\omega}{c_1}}}{2\sqrt{\omega}}, \qquad k_2 = \frac{\sqrt{1 - \frac{2\omega}{c_2}}}{2\sqrt{\omega}}.$$

If $c_1 < 0$ and $c_2 > 0$ or $c_1 > 0$ and $c_2 < 0$, the corresponding solution represents a combination of a single ω-soliton and a single ω-cuspon as $t \to \infty$.

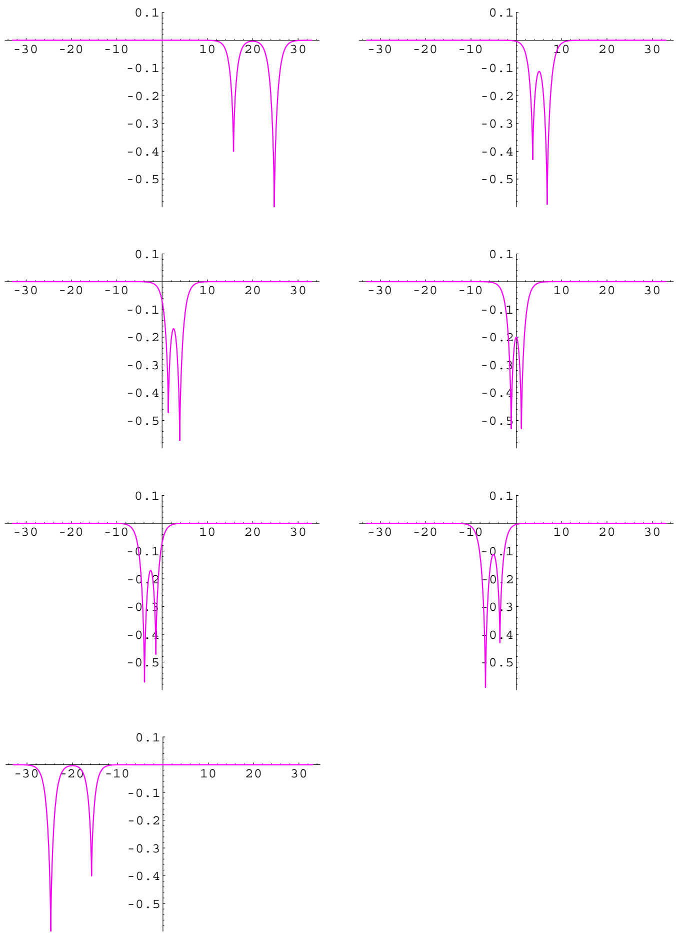

Figure 2. The interaction of two ω-cuspons. The velocities of two cuspons are, respectively, $c_1 = -0.4$ and $c_2 = -0.6$. From top to bottom, the time $t = (-40, -10), (-5, 0), (5, 10), 40$.

In figure 3, we have plotted the solution profiles for 11 different times, which describe the complete interaction process. Initially (at $t = -20$), the amplitude of the ω-soliton is $h_2 = c_2 - 2\omega = 0.498$, which is larger than the amplitude of the ω-cuspon that has the absolute value $h_1 = |c_1| = 0.4$. It can be seen that when the interaction begins both amplitudes start decreasing. At $t = -0.5$, the profile becomes one of complete elevation ($u > 0$ for all x). So, it is like that the ω-soliton (with a larger amplitude) 'eats up' the ω-cuspon (with a smaller amplitude). We also note that at all times there is always a point where $\partial_x u \to \infty$. This character of a cuspon is never destroyed by the interaction. At $t = 20$, the ω-soliton and ω-cuspon are well separated with their original amplitudes.

In figure 4, we represent another case of the interaction of an ω-cuspon and an ω-soliton. The former has an amplitude of $h_2 = c_2 = -0.6$ whose absolute value is larger than the amplitude of the latter that has the value of $h_1 = c_1 - 2\omega = 0.398$. When the interaction begins both amplitudes decrease. At $t = -0.6$, the 'larger' cuspon has 'eaten up' the smaller soliton. After $t > 0$, the soliton starts emerging again. Eventually at $t = 20$, the ω-soliton and ω-cuspon recover their original forms except with some phase shifts.

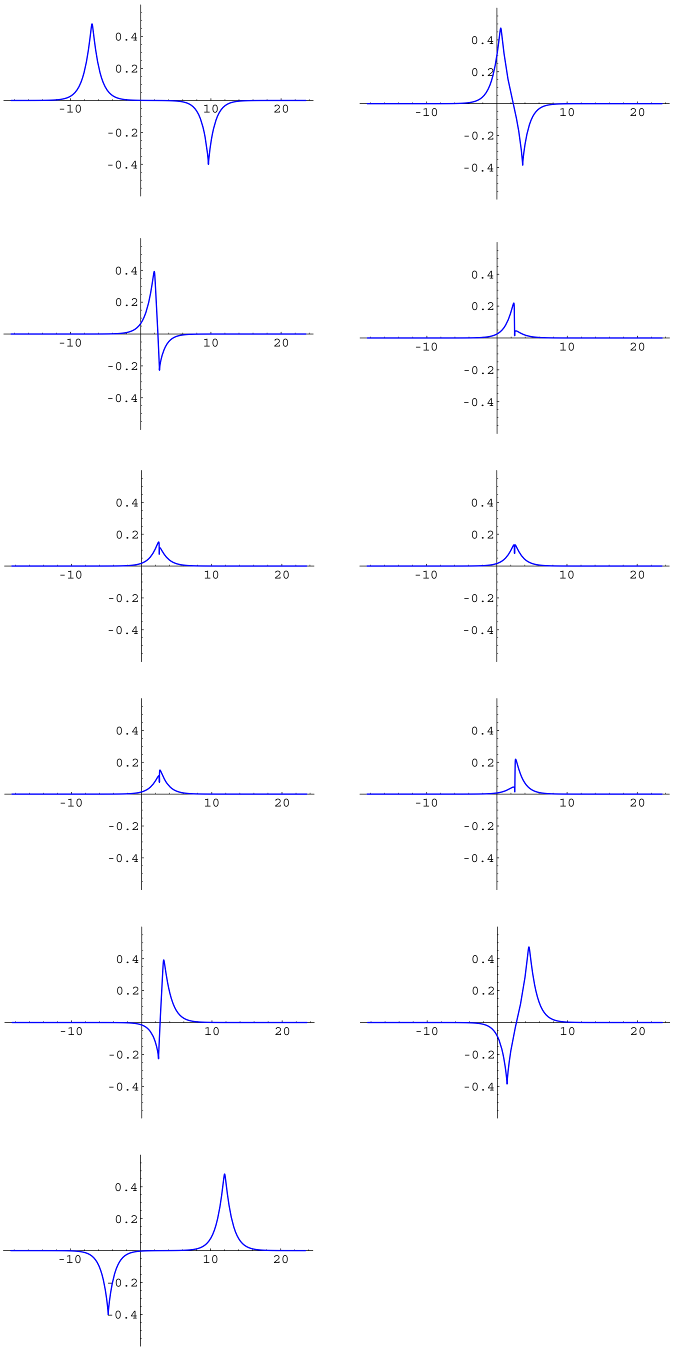

Figure 3. The interaction of an ω-soliton and an ω-cuspon for the case of $\omega = 0.01$. The velocities of the soliton and cuspon are, respectively, $c_1 = -0.4$ and $c_2 = 0.5$. From top to bottom, the time $t = (-20, -5)$, $(-2, -0.5)$, $(-0.1, 0)$, $(0.1, 0.5)$, $(2, 5)$, 20.

In figure 5, we also describe the case that the interaction of an ω-soliton and an ω-cuspon with almost equal amplitudes. More precisely, the ω-soliton has an amplitude $h_2 = c_2 - 2\omega = 0.398$ and the ω-cuspon has an amplitude $h_1 = c_1 = -0.46$. In this case, it is difficult to

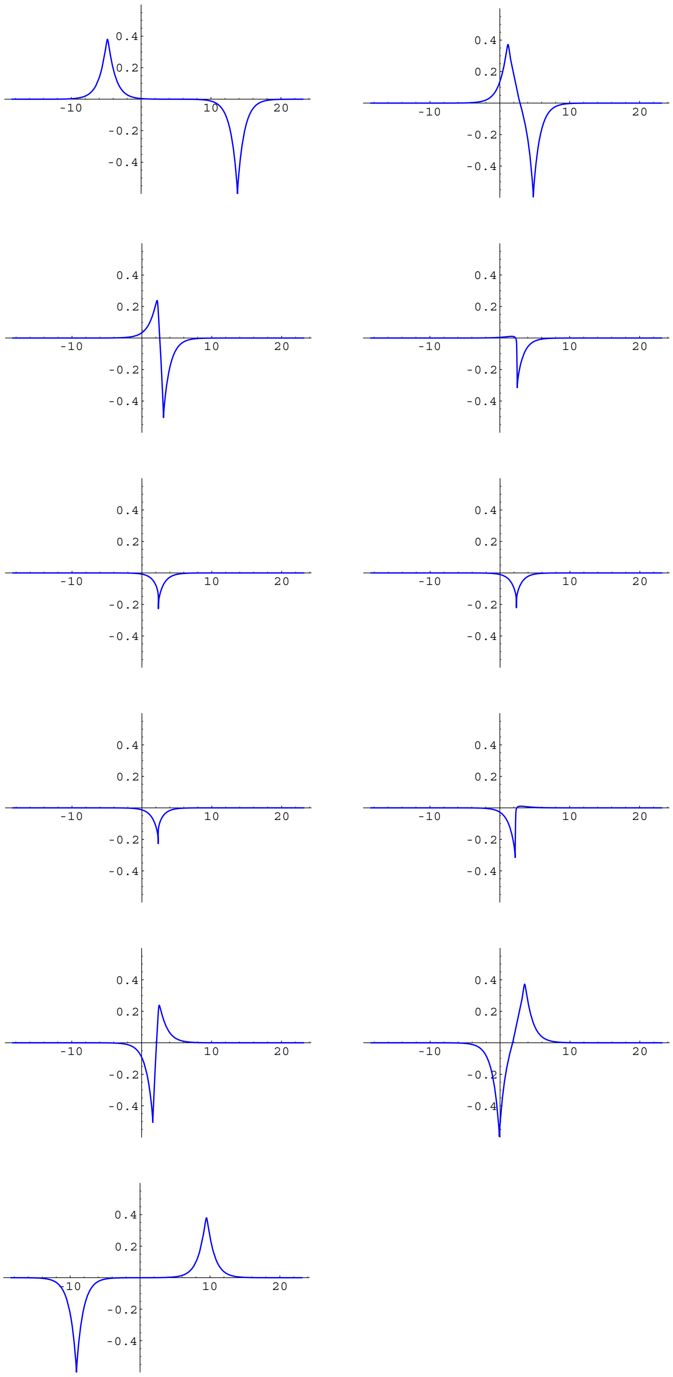

Figure 4. The interaction of an ω-soliton and an ω-cuspon for the case of $\omega = 0.01$. The velocities of the soliton and cuspon are, respectively, $c_1 = 0.4$ and $c_2 = -0.6$. From top to bottom, the time $t = (-20, -5), (-2, -0.6), (-0.1, 0), (0.1, 0.6), (2, 5), 20$.

distinguish who 'eats up' whom. Actually, at $t = 0$ the 'mass' $\int_{-\infty}^{+\infty} u \, \mathrm{d}x$ is almost equal to zero. But, still a soliton and a cuspon can emerge as time further evolves.

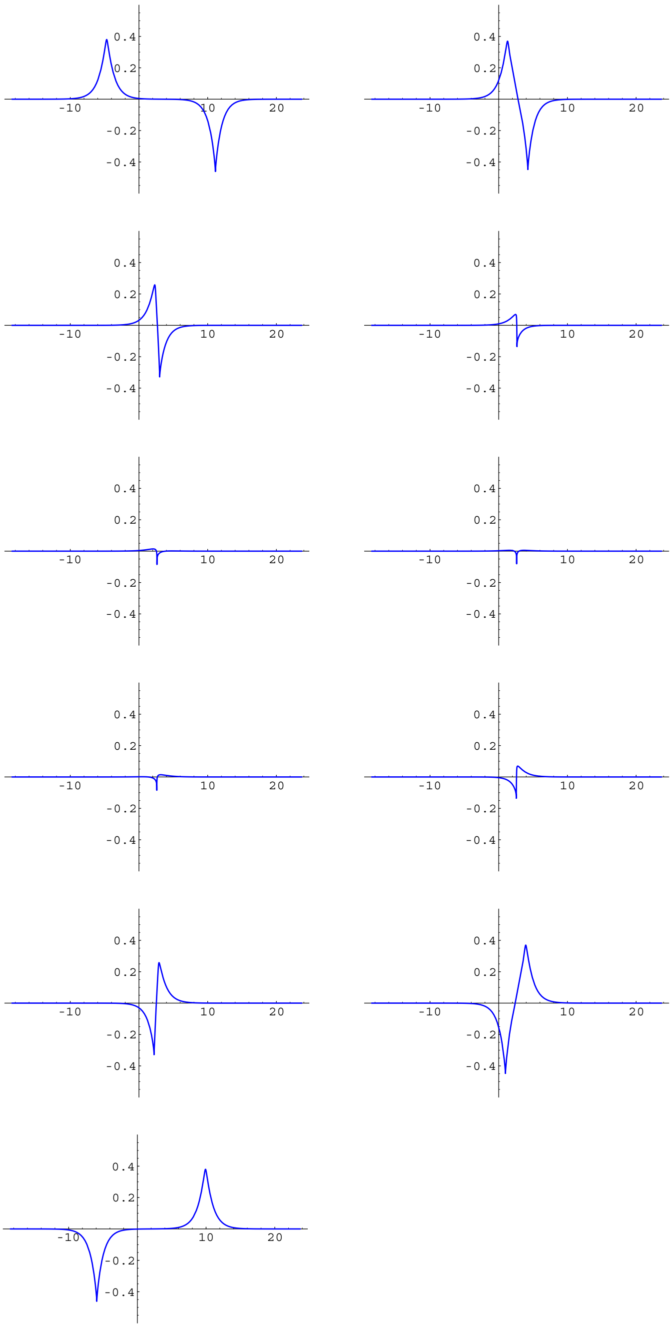

Figure 5. The interaction of an ω-soliton and an ω-cuspon for the case of $\omega = 0.01$. The velocities of the soliton and cuspon are, respectively, $c_1 = -0.46$ and $c_2 = 0.4$. From top to bottom, the time $t = (-20, -5)$, $(-2, -0.6)$, $(-0.1, 0)$, $(0.1, 0.6)$, $(2, 5)$, 20.

Finally, we point out that the phase shifts after the interactions of two solitons have been given in Johnson (2003) and Li (2005) and the phase shifts after the interaction of a soliton and a cuspon have been given in Ferreira *et al* (1999). To get the phase shifts after the interactions among n solitons and m cuspons is a difficult task, and we shall leave this for a future work.

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